

# The Harmonic Balance Method

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**Abstract**—The harmonic balance method is used in most current commercial RF simulation tools. This is due to the fact that it has certain advantages over other common methods used in SPICE, namely (modified) nodal analysis (MNA), which make it better suited to simulate stiff problems and circuits containing transmission lines, nonlinearities and dispersive effects.

In this project, the harmonic balance algorithm is derived and explained; a basic harmonic balance solver is developed and implemented in Matlab and applied to some basic linear and nonlinear circuits.

**Index Terms**—harmonic balance, nonlinear circuit, rf circuit

## I. INTRODUCTION

THE harmonic balance (HB) method is probably the most common technique to simulate microwave and RF circuits, since it can efficiently account for transmission line and nonlinear effects, operating in a periodic or quasi-periodic steady-state regime. It is known to be fast, efficient and versatile. With it being a (hybrid) frequency-domain method, dispersive effects can also be included and it commonly the method of choice for the optimization of circuits and circuit parameters [5], [10].

The HB method is used in the industry’s leading microwave and RF simulation tools, such as Agilent’s ADS/Genesys, AWR Microwave Office, Ansoft’s Designer/Nexxim and Cadence Virtuoso Spectre.

Nowadays, when people talk about harmonic balance methods, what is meant is what was originally known as piecewise harmonic balance method [1], thus in this project only this version will be considered.

### A. Time-Domain Analysis

The modified nodal analysis (MNA) method [2], most commonly used by SPICE-like circuit simulators, is a well-known and powerful technique to form a linear matrix equation system of the form

$$\mathbf{G}\mathbf{x}(t) + \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{B}\mathbf{u}(t) \quad (1)$$

from a given circuit in the time-domain. The term *modified* refers to the fact that  $\mathbf{x}$  does not contain only node voltages but also certain currents, to overcome the shortcomings of the standard nodal analysis method and be able to simulate ideal voltage sources and inductors.

Thus, MNA and time-domain solutions in general are able to simulate circuits containing all kinds of lumped circuit elements—but it has some shortcomings: [3]–[5]

- extending transient analysis out to steady-state is computationally inefficient
- issues with stiff problems (poor convergence, long simulation time, etc.)
- dispersion is hard to implement (since frequency-domain described models cannot be implemented)
- not effective for optimization and statistical analysis.

These weaknesses only appear in special cases, including microwave circuits (especially those including transmission line effects) resulting in stiff<sup>1</sup> problems (where usually several periods, e.g. 10 to 30 [3] periods are necessary for steady-state approximations; for high- $Q$  problems even many more), switched capacitor and crystal/cavity filters and multitone excitation.

### B. Frequency-Domain Analysis

Frequency-domain methods usually solve for steady-state solutions by inverting a matrix equation system similar to

$$[\mathbf{G} + j\omega\mathbf{C}]\mathbf{x}(\omega) = \mathbf{B}\mathbf{u}(\omega) \quad (2)$$

where each the solutions are obtained for each point in the frequency directly. For a small number of frequencies this can therefore be considerably more efficient than using time-domain methods, since the linear dynamic operations such as differentiation and integration become simple algebraic operations [5]. However, with the standard principle only solutions of linear time-invariant circuits can be obtained.

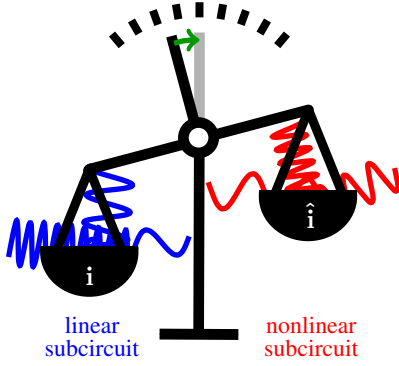
### C. Harmonic Balance – A Hybrid Method

The harmonic balance method is a hybrid time- and frequency-domain approach, which allows all the advantages of nonlinear time-domain device modelling, combined with the strength (namely efficiency) of the steady-state frequency-domain technique.

The name stems from the idea of balancing the currents obtained from the nonlinear subcircuits (via time-domain) and the currents from the linear subcircuits at the subcircuit interfaces, in the frequency-domain<sup>2</sup>, as illustrated in Fig. 1.

<sup>1</sup>As discussed in the lectures, powerful model-order reduction (MOR) techniques were developed to overcome these problems by approximating only parts of interest. However, in this project only full-order problems (not approximated) are considered.

<sup>2</sup>The balancing can also be done in the time domain, which is usually called time-domain harmonic balancing or waveform balancing [5], which however is not considered any further, here.



**Fig. 1:** Illustration of the harmonic balance principle: the algorithm tries to balance the currents (harmonics) from the linear and nonlinear subcircuits in the frequency domain by iteratively comparing them and applying adequate changes.

#### D. Alternatives

There are only a few alternatives to time-domain transient methods (possibly using a prohibitively small time step) and the harmonic balance method, for example:

- Volterra series [6], which decompose the nonlinearity into a weighted power series. This is only suitable for weakly nonlinear problems, where only a few higher-order terms are sufficient to describe all effects.
- Shooting methods [3], where the transient response is bypassed by iteratively choosing the initial conditions to achieve steady state. Due to efficiency considerations, this is only suitable for periodic problems with a small period.

#### E. Notation

Throughout this document italic letters refer to scalar variables, small bold letters to vectors and capital bold letters to matrices, *everywhere*. For the voltages and currents, the same symbols are used for time and frequency domain, i.e.  $\mathbf{v}$ ,  $\mathbf{i}$  (and  $\mathbf{x}$  as in Eqns. (1) and (2)); the distinction will either become apparent from the context or the respective domain is specified in brackets, whenever necessary. The quantities resulting from nonlinearities are set with a hat, e.g.  $\hat{\mathbf{i}}$ , to separate them from the linear quantities (appearing at the same node).

## II. BALANCING THE HARMONICS

### A. Preliminaries

HB uses the assumption that the total voltage can be (approximately, but accurately enough) composed of the finite Fourier series

$$v(t) = \sum_{k=-K+1}^K v(k) e^{jk\omega_0 t} \quad \forall K \in 2^n, n \in \mathbb{N} \quad (3)$$

with the phasors  $v(k) \in \mathbb{C}$ . Thus, the signal is periodic

$$v(t+T) = v(t) \quad (4)$$

with period<sup>3</sup>  $T = 2\pi/\omega_0$ . In the single-tone cases with simple nonlinearities it is usually assumed that the excitation

frequency  $\omega_s$  is the lowest frequency in the system (apart from DC) and thus, usually  $\omega_0 = \omega_s$  is used. However,  $\omega_0$  could also be chosen as  $1/2$  or  $1/4\omega_s$ , etc. In this project report the following notation will be used for the discrete Fourier transform (DFT, FFT):

$$\mathbf{v}(\omega) = \mathcal{F}\mathbf{v}(t) \quad (5)$$

where using the *twiddle factor*  $w = e^{-j2\pi/K}$  the DFT matrix of dimension  $K \times K$  is

$$\mathcal{F} = \frac{1}{K} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^k & \dots & w^{K-1} \\ 1 & w^2 & w^4 & \dots & w^{2k} & \dots & w^{2(K-1)} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & w^k & w^{2k} & \dots & w^{k^2} & \dots & w^{k(K-1)} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & w^{K-1} & w^{2(K-1)} & \dots & w^{k(K-1)} & \dots & w^{(K-1)^2} \end{bmatrix}$$

and similarly  $\mathbf{v}(t) = \mathcal{F}^{-1}\mathbf{v}(\omega)$ . Further, using the fact that

$$\mathcal{F}^{-1} = 2K\mathcal{F}^* \quad (6)$$

$\mathcal{F}$  has never actually to be inverted. Note that only for this derivation the DFT is stated in matrix form; in the later implementation the FFT algorithm is used.

Assuming that the output signal is real, we know that its spectrum is Hermitian; thus we only have to consider  $K+1$  frequencies ( $K$  harmonics and DC), where for maximum efficiency of the FFT algorithm  $K$  should be chosen as  $2^n$  with  $n \in \mathbb{N}$ .

### B. Nonlinear Systems

The most general form of equation to model nonlinear circuits is [11]

$$\frac{d}{dt} \mathbf{q}(\mathbf{v}) + \mathbf{i}(\mathbf{v}) + \mathbf{u} = 0 \quad (7)$$

where the fact that  $\mathbf{v} = \mathbf{v}(t)$  and  $\mathbf{u} = \mathbf{u}(t)$  is omitted to keep the equation less crowded. Note that both  $\mathbf{q}(\cdot)$  and  $\mathbf{i}(\cdot)$  can be nonlinear functions; the first maps the vector  $\mathbf{v}$  consisting of mostly voltage potentials to a vector whose entries are mostly sums of capacitive charges or inductive fluxes and the latter mostly accounts for conductive and nonlinear source effects.

If  $\mathbf{q}(\cdot)$  is linear (i.e. there are no nonlinear capacitive or inductive effects<sup>4</sup>) and  $\mathbf{i}(\mathbf{v}) = \mathbf{G}\mathbf{v} + \mathbf{g}(\mathbf{v})$  can be split in a linear and a nonlinear part this results in the nonlinear equivalent of Eqn. (1):

$$\mathbf{C}\dot{\mathbf{v}}(t) + \mathbf{G}\mathbf{v}(t) + \mathbf{g}(\mathbf{v}(t)) + \mathbf{u}(t) = 0 \quad (8)$$

Within this project only this type of nonlinear systems will be considered.

<sup>4</sup>Nonlinear inductive effects are rarely used in microwave circuits [10] but nonlinear capacitors are commonly found, mostly in form of varactors.

<sup>3</sup>There are also formulations for almost periodic functions, see [9], [10].

### C. Derivation

The following brief derivation focuses on the circuit shown in Fig. 2 but can easily be generalized. The derivation is necessary since the formulas in [6] are incomplete, seem to contain some errors and did not lead to a useful algorithm. The underlying idea is similar, but follows more the one in [10], with a different attempt at the Jacobian.

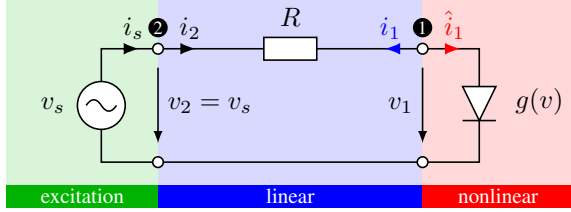


Fig. 2: Simple circuit consisting of three parts.

The currents from the nonlinear and linear subcircuits (sides) at node 1 have to be balanced, meaning the same (but of opposite sign), in for all harmonics in the frequency domain  $k \in [0, \dots, K]$ :

$$i_1(k\omega_0) + \hat{i}_1(k\omega_0) = 0 \quad \forall k \Leftrightarrow \mathbf{i}_1(\omega) + \hat{\mathbf{i}}_1(\omega) = \mathbf{0} \quad (9)$$

1) *Linear subcircuit*: The linear current is given by

$$\mathbf{i}_1(\omega) = \underbrace{\mathbf{Y}_2 \mathbf{v}_2}_{\mathbf{i}_2(\omega)} + \mathbf{Y}_1 \mathbf{v}_1 = \underbrace{\mathbf{Y}_s \mathbf{v}_s}_{\mathbf{i}_s(\omega)} + \mathbf{Y} \mathbf{v}_1. \quad (10)$$

and since the node number 1 is the only remaining node, this specification will be omitted to simplify notation. Thus stands

$$\mathbf{i}(\omega) = \mathbf{Y}_s \mathbf{v}_s + \mathbf{Y} \mathbf{v} = \mathbf{i}_s(\omega) + \mathbf{i}_{\text{lin}}(\omega). \quad (11)$$

The source components are

$$\mathbf{i}_s(\omega) = \mathbf{Y}_s \mathbf{v}_s = \begin{bmatrix} Y_{12}(0) & & & \\ & Y_{12}(\omega_0) & & \\ & & \ddots & \\ & & & Y_{12}(k\omega_0) \\ & & & & \ddots \\ & & & & & Y_{12}(K\omega_0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ = \text{Diag}[\mathbf{y}_{12}(\omega)] \mathbf{v}_s \quad (12)$$

and the components due to the interface voltage  $v_1$  are, similarly,

$$\mathbf{i}_{\text{lin}}(\omega) = \mathbf{Y} \mathbf{v} = \text{Diag}[\mathbf{y}_{11}(\omega)] \mathbf{v} \quad (13)$$

where  $Y_{11}$  and  $Y_{12}$  are the respective (generally frequency-dependend) entries of the admittance matrix of the linear subcircuit (in the frequency domain, thus a vector of  $K+1$  entries).  $\text{Diag}[\cdot]$  is an operator putting the input vector in diagonal matrix form. And the vector of unknowns in the frequency domain is

$$\mathbf{v} = [v(0), v(\omega_0), v(2\omega_0), \dots, v(k\omega_0), \dots, v(K\omega_0)]^T \quad (14)$$

2) *Nonlinear subcircuit*: The nonlinear current  $\hat{\mathbf{i}}(\omega)$  is obtained by taking the inverse Fourier transform of the (presumably not yet correct) node voltage in the frequency domain, applying the nonlinearity in the time domain, and transforming back to the frequency domain. Note that for this procedure, the voltage vector is used in full (not just  $K$  entries) using the conjugate-complex values at the right spots and thus  $2K$  time samples are obtained:

$$\mathbf{v}(t) = \mathcal{F}^{-1} \begin{bmatrix} \mathbf{v}(\omega) \\ v_{K-1}^*(\omega) \\ \vdots \\ v_1^*(\omega) \end{bmatrix} \quad \begin{array}{l} \mathbf{v}(t) \in \mathbb{R}^{2K} \\ \mathbf{v}(\omega) \in \mathbb{C}^{K+1} \end{array} \quad (15)$$

3) *The algorithm*: The HB algorithm works as illustrated in Fig. 3, namely:

- 0) Initial guess  $\mathbf{v}^0(t)$  (could also be in frequency domain). Most commonly a zero initial value is used, however also damped versions of the excitation and impulses (or noise, with broad spectral content) can be good start values<sup>5</sup>.
- 1) Apply nonlinearity in the time domain, either via nonlinear conductivity  $\hat{i}(t) = v(t)g(v)$  or directly as voltage-dependent current as for the diode current in this case:

$$\hat{i}(v) = \hat{i}_d(v(t)) = I_s \cdot (e^{v/v_T} - 1) \quad (16)$$

in the actual vector form

$$\hat{\mathbf{i}}^m(t) = \hat{\mathbf{i}}^m(\mathbf{v}^m(t)) \quad (17a)$$

$$= \hat{\mathbf{i}}_d(\mathbf{v}^m(t)) = \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}^m(\omega)) \quad (17b)$$

- 2) Go back to frequency domain

$$\hat{\mathbf{i}}^m(\omega) = \mathcal{F} \hat{\mathbf{i}}^m(t) = \mathcal{F} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}^m(\omega)) \quad (18)$$

- 3) Check if the harmonics are balanced using the cost function in the frequency domain

$$\mathbf{f}(\mathbf{v}^m) = \mathbf{i}^m(\omega) + \hat{\mathbf{i}}^m(\omega) \quad (19a)$$

$$= \mathbf{Y}_s \mathbf{v}_s + \mathbf{Y} \mathbf{v}^m + \mathcal{F} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}^m) \quad (19b)$$

<sup>5</sup>Especially when considering oscillators (without excitations), as shown later, zero initial guess leads to wrong (or rather the trivial zero) solutions

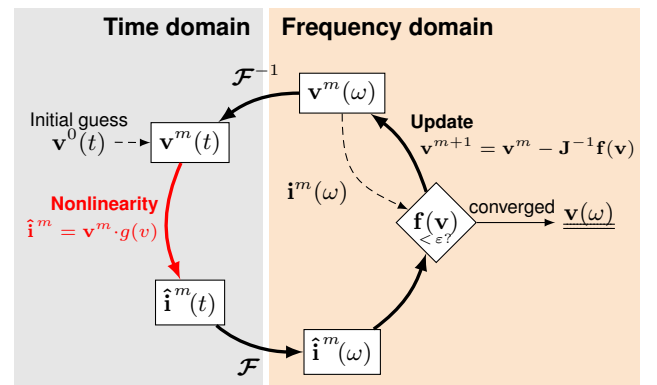


Fig. 3: Algorithm working principle.

- 4) If the convergence criteria were not satisfied, updated the voltages using the cost function and its Jacobian (still in the frequency domain):

$$\mathbf{v}^{m+1} = \mathbf{v}^m - \mathbf{J}^{-1} \mathbf{f}(\mathbf{v}^m) \quad (20)$$

- 5) Start the process over, by going to the time domain, to apply the nonlinearity once more.

The main problem is finding the Jacobian:

$$\mathbf{J} = \left. \frac{d\mathbf{f}(\mathbf{v})}{d\mathbf{v}} \right|_{\mathbf{v}=\mathbf{v}^m(\omega)} \Leftrightarrow J_{ij} = \frac{\partial f_i(\mathbf{v})}{\partial v_j}$$

Following the previous notation, the Jacobian can be found column-wise according to (for the  $j$ th column)

$$\mathbf{J}_j = \frac{\partial \mathbf{f}(\mathbf{v})}{\partial v_j} = \mathbf{Y}_j + \frac{\partial \hat{\mathbf{i}}(\omega)}{\partial v_j}. \quad (21)$$

The derivative of the nonlinear current can be found using in the definition from Eqn. (17b)

$$\frac{\partial}{\partial v_j} [\mathcal{F} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}(\omega))] = \mathcal{F} \frac{\partial}{\partial v_j} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}(\omega)) \quad (22)$$

where the place holder  $\tau = \mathcal{F}^{-1} \mathbf{v}(\omega)$  is introduced to solve the partial derivative of the nonlinear diode current according to

$$\frac{\partial}{\partial v_j} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}(\omega)) = \left. \frac{\partial \hat{\mathbf{i}}_d(\tau)}{\partial \tau} \right|_{\tau=\mathcal{F}^{-1} \mathbf{v}} \frac{\partial \tau}{\partial v_j} \quad (23a)$$

$$= \hat{\mathbf{i}}_d'(\mathcal{F}^{-1} \mathbf{v}(\omega)) * \mathcal{F}_j^{-1} \quad (23b)$$

where  $\mathcal{F}_j^{-1}$  is the  $j$ th column of the IDFT matrix and the asterisk denotes a point-wise multiplication (*not* a convolution). The derivative of the current can be found analytically in this case, since the diode current is known in form of a simple exponential function; thus  $\hat{\mathbf{i}}_d'$  can remain in the result. A better notation is forming a diagonal matrix, to multiply the vector with:

$$\frac{\partial}{\partial v_j} \hat{\mathbf{i}}_d(\mathcal{F}^{-1} \mathbf{v}(\omega)) = \text{Diag}(\hat{\mathbf{i}}_d'(\mathcal{F}^{-1} \mathbf{v}(\omega))) \mathcal{F}_j^{-1}. \quad (24)$$

After this, common matrix multiplication can be applied which leads to the same result. Thus, one-by-one (for each column  $j$  independently) the columns of the Jacobian are obtained, leading to the total matrix

$$\mathbf{J} = \mathbf{Y} + \mathcal{F} \text{Diag}(\hat{\mathbf{i}}_d'(\mathcal{F}^{-1} \mathbf{v}(\omega))) \mathcal{F}^{-1} \quad (25)$$

where we only the  $j$  specifying the respective row of the IDFT matrix had to be removed. Note that this result derivation differs<sup>6</sup> from the one in [6].

As stated before, we do not have to actually invert the DFT matrix but can use the conjugate-complex instead. Also, since the dimension of the time-domain values is  $\mathbf{v}(t) \in \mathbb{R}^{2K}$  but the one of the frequency-domain values is  $\mathbf{v}(\omega) \in \mathbb{C}^{K+1}$ , the DFT matrix is of (rectangular) size  $K+1 \times 2K$  and the inverse turns out to be not just the conjugate-complex but the Hermitian. Thus, the implemented Jacobian is finally

$$\mathbf{J} = \mathbf{Y} + 2K \mathcal{F} \left[ \text{Diag}(\hat{\mathbf{i}}_d'(2K \mathcal{F}^H \mathbf{v}(\omega))) \mathcal{F}^H \right] \quad (26)$$

and is of size  $\mathbb{C}^{K+1 \times K+1}$ .

<sup>6</sup>There appear to be some mistakes in the derivation in appendix A.6.

#### D. Implementation of the algorithm

The next few subsections express some details of the implementation of the algorithm in Matlab.

1) *Nonlinearity stamp*: In this project the MNA code from previous assignments was adapted to now also account for (single-port) nonlinearities. The new command is defined as

$$\text{Nvalue node+ node-}$$

where as usually `value` can be any value to distinguish the nonlinearity and `node+` and `node-` are the anode and cathode nodes, respectively.

The MNA code then interprets this as a voltage source (thereby also adding the current through it to the vector of unknown variables) with unit voltage. Further, it produces the input matrix  $\mathbf{B} \in \mathbb{R}^{N \times N_s}$ , where  $N$  is the total number of variables and  $N_s$  is the number of sources and nonlinearities, along with the source vector  $\mathbf{u} \in \mathbb{C}^{N_s}$ , as shown in Eqns. (1) and (2), whereas previously a source vector  $\mathbf{b} = \mathbf{B}\mathbf{u} \in \mathbb{R}^N$  was used, directly.

2) *Admittance matrices*: With the input matrix  $\mathbf{B}$  and the source vector  $\mathbf{u}$  separate, the way to get the required source and nonlinearity admittance matrices is straightforward: According to the definition, the (self-) admittance at (nonlinear) port 1 and the admittance as seen from the source are:

$$Y_{11} = \left. \frac{i_1}{v_1} \right|_{v_n=0, \forall n \neq 1} \quad Y_{s1} = \left. \frac{i_1}{v_s} \right|_{v_n=0, \forall n \neq 1} \quad (27)$$

Since all  $v_n$  are either source or nonlinear port voltages and thus accounted for in the source vector  $\mathbf{u}$ , the matrix entries can be obtained according to, for example

$$\mathbf{v} = (\mathbf{G} + jk\omega_0 \mathbf{C})^{-1} \mathbf{B}\mathbf{u}_1 \quad (28)$$

$$Y_{11}(k\omega_0) = \frac{i_{\text{NL1}}}{u_{\text{NL1}}} \xrightarrow{u_{\text{NL1}}=1} i_{\text{NL1}} \quad (29)$$

where  $\mathbf{u}_n$  refers to an excitation vector with only entry  $n$  (for port  $n$ ) equal to 1, thus exciting the system only at that one port (with unit amplitude) and  $i_{\text{NL1}} \in \mathbf{v}$  is an entry in the vector of unknowns  $\mathbf{v}$  (consisting of the voltages of the regular and currents of the modified MNA equations). The way the new `NodalAnalysis()` script is set up, the nonlinear currents are always at the end of the vector of unknowns  $\mathbf{v}$ .

While this way a very general code results, there are some shortcomings: whenever the nonlinearities of the circuit to be simulated lie (almost) in parallel to one another, by shorting one out, others get shorted out as well and the admittance matrix only has (almost) zero entries. Thus, these circuits, e.g. full-wave rectifier, can not be simulated with the current implementation.

3) *Source stepping*: Source stepping proved to increase the rate of convergence of nonlinear problems for the MNA method, before. The same is true for HB — at an even greater extent. Some problems, especially involving strong nonlinearities (e.g. diodes) along with dynamic elements (capacitors and/or inductors) do not converge without it, but instead the solution becomes unstable and blows up or leads to singular matrices.

In this project the simplest way of source stepping was implemented. Since the source is only contained in the cost

function, the source stepping variable  $\lambda$  is used there, leading to

$$\mathbf{f}(\mathbf{v}) = \lambda^m \mathbf{Y}_s \mathbf{v}_s + \mathbf{Y} \mathbf{v}^m + \mathcal{F} \mathbf{i}_d(\mathcal{F}^{-1} \mathbf{v}^m) \quad (30)$$

where the initial value in the range of  $\lambda^0 \in [0.01, 0.25]$  was most commonly used and the update is  $\lambda^{m+1} = \lambda^m + \Delta\lambda^m$

$$\Delta\lambda^{m+1} = \begin{cases} 2\Delta\lambda^m & \text{if } \mathbf{v} \text{ converged in } M \text{ steps} \\ \frac{1}{2} \Delta\lambda^m & \text{if } \mathbf{v} \text{ did not converge in } M \text{ steps} \end{cases} \quad (31)$$

where the initial update  $\Delta\lambda^0$  was usually chosen in the range  $\Delta\lambda^0 \in [0.01, 0.05]$ . Using source stepping proved most effective with zero initial values;  $\mathbf{v}^0 = 0$ . For fair comparisons the same was implemented in the MNA method.

4) *Speed-up using FFT*: As mentioned during the derivation, implementing the algorithm using the DFT in matrix form is not efficient, especially when a large number of harmonics is looked for. The matrix multiplications from left can simply be replaced with the respective FFT/IFFT functions<sup>7</sup>.

The multiplication from right with the Hermitian transpose  $\mathcal{F}^H$  from right can be done using the Matlab commands

$$2K\mathcal{F} \left[ \text{Diag}(\mathbf{i}'_d(\mathbf{v}(t))) \mathcal{F}^H \right] = \text{ifft}(\text{fft}(\text{diag}(\text{did})) \cdot') \cdot' \quad (32)$$

where  $\cdot'$  stands for the (normal) transpose and  $\text{did}$  is the differentiated diode current vector in the time domain,  $\hat{\mathbf{i}}'_d(t)$ , consisting of  $2K$  samples. Note that this result (despite the fact that an IFFT was used last) is in the frequency domain and that only a (square) submatrix ranging over the entries from indices 1 to  $K+1$  are used (since the negative side of the spectrum is the complex conjugate of the positive side).

5) *Complex variables*: Normally, the variables have to be split into real and imaginary parts [10], thereby almost doubling the size of the problem. However, since this algorithm was implemented in Matlab, which can work with complex variables directly, this was not done here.

Additionally, it is common to zero out [10] the imaginary part of the DC component in the spectrum (since it is expected to be zero in the end anyway) or to not consider it at all, when the variables are split up. Personal experience shows that this in fact *can* help the rate of convergence for some circuits, but also leads to more severe problems for others. Thus, if convergence is not achieved or achieved too slowly, zeroing/not zeroing out the imaginary part of the DC could help.

### III. NONLINEAR CIRCUITS

Essentially, with nonlinear circuits, the transfer characteristic is a function of the signal level, in addition to all the other parameters for linear circuits. As Stephen Maas, well-known researcher, author and now director of technology at AWR, writes in [10]

*"All electronic circuits are nonlinear: this is a fundamental truth of electronic engineering."*

<sup>7</sup>Note that the `fft()` and `ifft()` commands in Matlab use a different normalization than can be expected; thus the FFT result is divided by the length of the vector, whereas the IFFT is multiplied by it.

Still, it is not necessarily easy to find a good example for nonlinear and stiff circuits. The traditional way to show nonlinearities is to show the generation of new frequencies using a power series with multi-mode excitation, compare spectra, etc. [5], [10]

In the following the use and performance of the harmonic balance algorithm will be discussed for the following basic nonlinear circuits:

- Half-wave rectifier with filter capacitor
- Half-wave doubler (to show multiple nonlinearities)
- Mixer diode (to show effects of multiple excitations)
- Van der Pol oscillator (to show a different kind of nonlinearity)

Clearly, this list could be extended almost indefinitely, but it is believed that the most important characteristics of the harmonic balance methods can be seen from these examples and thus no further circuits were considered.

#### A. Half-wave rectifier with capacitor

The first example, as shown in Fig. 4, is basically the same as used in the derivation of the harmonic balance method (see Fig. 2), only that in this case the diode is "floating" (i.e. has two non-zero nodes) and an additional dynamic element in form of the output  $RC$  filter has been added.

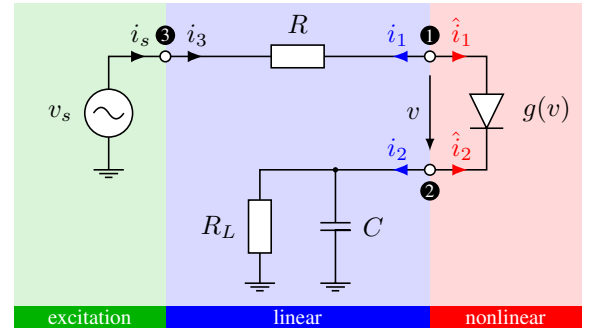


Fig. 4: Half-wave rectifier with capacitor and load.

Fig. 5 shows the expected result in the time-domain: depending on the time constant of the output  $RC$  circuit, it takes a large number of periods for the output voltage ( $v_2$ , blue) to reach steady-state (approximately, e.g. with an overall  $L_2$  error  $< 10^{-6}$ ).

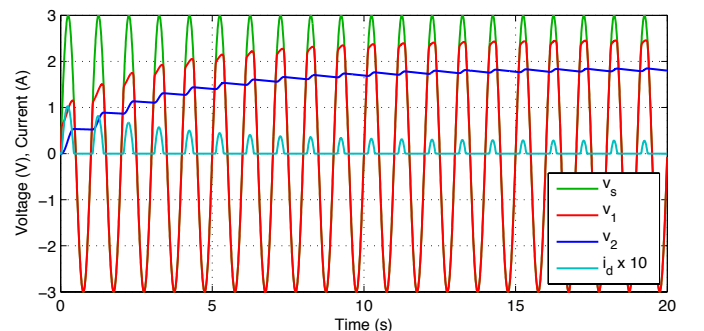
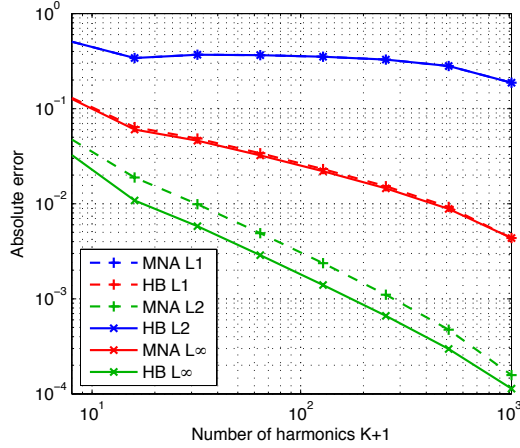


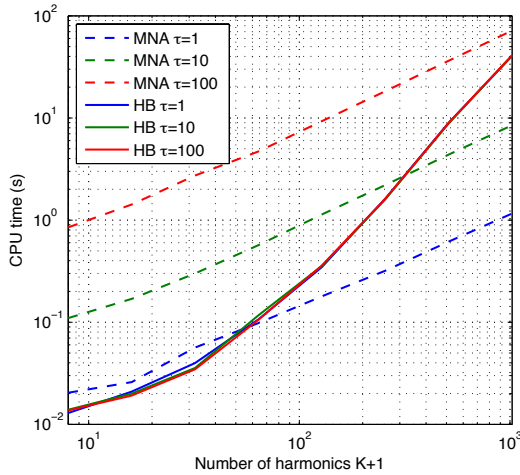
Fig. 5: Time domain (MNA) result of the filtered half-wave rectifier circuit.

While the errors are generally comparable (see Fig. 6), the comparison of the computational efficiency leads to the expected conclusion: the harmonic balance method is much more efficient in calculating the steady-state of this circuit. While the CPU time of the MNA solution increases with



**Fig. 6:** Comparison of the absolute errors of HB and MNA vs. number of harmonics  $K + 1$ , for various time constants  $\tau = R_L C$ .

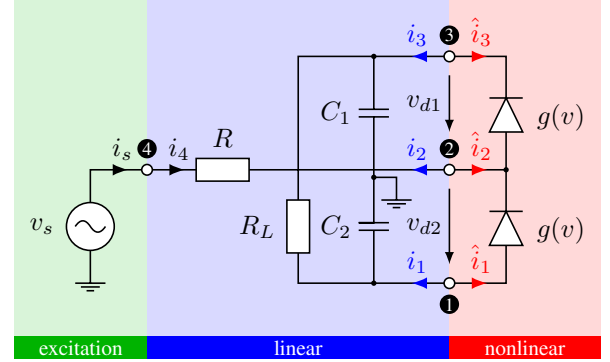
increasing time constant  $\tau$  (slower dynamic of the output, therefore longer time until the steady-state is reached), it stays (almost) constant for HB. Thus, as long as the total number of harmonics can be held small (i.e. for example lower than 64), then HB is (much) more efficient. For large numbers of harmonics, eventually the computational demands of the HB increase faster than for the MNA. This makes sense, since for the time-domain method to achieve more harmonics only some more samples in the time-domain have to be calculated (and then Fourier transformed), whereas for the HB the impedance matrices increase for each additional harmonic. Thus, for large numbers of harmonics (i.e. broadband simulations), the HB should probably not be the method of choice. Also, if the dynamics are very fast, the computational demands are comparable.



**Fig. 7:** CPU time consumption vs. number of harmonics  $K + 1$ , for various  $RC$  values.

### B. Half-wave doubler rectifier

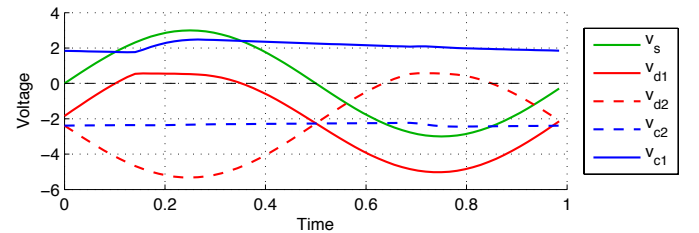
The Delon bridge voltage doubler consists of two diodes and two capacitors, as shown in the schematic in Fig. 8. To show that the simulator can handle two nonlinearities totally independently from one another and with different dynamics, the capacitor values were chosen differently:  $C_2 = 4C_1 = 2 \text{ mF}$ ,  $R_L = 10 \text{ k}\Omega$  and  $R = 3 \Omega$ ,  $v_{s \text{ peak}} = 3 \text{ V}$  (1 Hz).



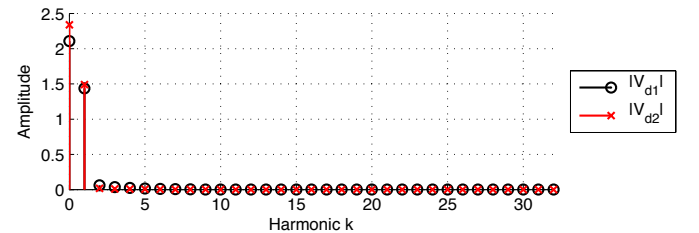
**Fig. 8:** Delon bridge voltage doubler circuit.

The time- and frequency-domain plots in Fig. 9 show the expected behavior: the dynamics in the upper and lower part of the rectifier differ, with the voltage at the upper capacitor  $C_1$  dropping more than the one of capacitor  $C_2$  (since it has a larger capacitance and thus a greater filter effect).

A similar result can be seen in the spectrum: the DC-offset is larger at the second diode, while the first harmonic is of a comparable level.



(a) Solution in time domain



(b) Spectrum of the solution

**Fig. 9:** Solution of the Delon bridge voltage doubler.

### C. Diode as mixer

As commonly used in microwave and RF circuits, due to its nonlinearity, a diode can serve as frequency mixer elements. Fig. 10 shows the basic diode mixer schematic, where two excitations act on a single nonlinearity.



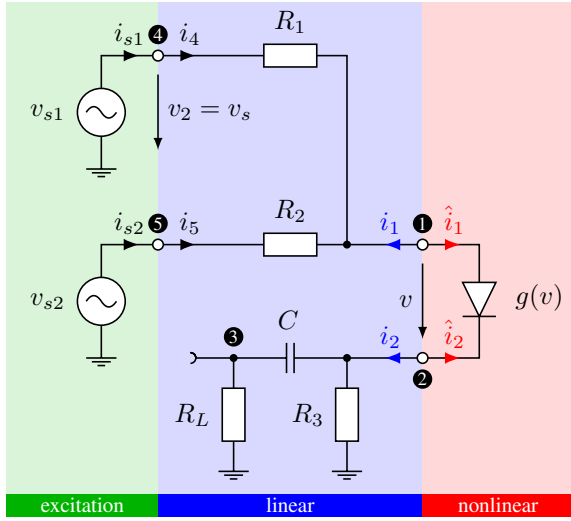


Fig. 10: Two-tone diode mixer circuit.

The Taylor expansion of the diode current function reveals the power series

$$i_d(v) = I_s(e^{v/v_T} - 1) = \left( \frac{v}{v_T} + \frac{v^2}{v_T^2} + \frac{v^3}{v_T^3} + \dots \right). \quad (33)$$

Thus, when the voltage consists of two cosines of different frequencies

$$v = \cos \omega_1 + \cos \omega_2 \quad (34)$$

then because of the nonlinear conductance behavior of the diode, besides the excitation frequencies, other frequencies result. From the second- and third-order terms follow:

$$\begin{aligned} v^2 &= 1 + \frac{\cos 2\omega_1 t + \cos 2\omega_2 t}{2} + \cos(\omega_1 t \pm \omega_2 t) \\ v^3 &= \frac{9}{4}(\cos \omega_1 t + \cos \omega_2 t) + \frac{1}{4}(\cos 3\omega_1 t + \cos 3\omega_2 t) \\ &\quad + \frac{3}{4}(\cos(2\omega_1 t \pm \omega_2 t) + \cos(2\omega_2 t \pm \omega_1 t)) \end{aligned}$$

This is confirmed by the harmonic balance simulation, where the excitation frequencies were  $\omega_1 = 7\omega_0$  and  $\omega_2 = 11\omega_0$  and the second-order products  $11+7 = 18$  and  $11-7 = 4$  as well as  $2 \times 11 = 22$  and  $2 \times 7 = 14$  can be seen right away. The third-order products are already much small in amplitude, but together with higher-order terms form a more "noisy" spectrum around the dominant modes. As can also be seen, the output high-pass filter does its job well: there is no DC in the output signal.

Multi-tone simulations with nonlinearities generally lead to many non-zero values spread over a wide band of the spectrum, thus the number of harmonics  $K$  has to be chosen high enough, in order to avoid aliasing effects of frequencies which cannot be resolved correctly.

The primary problem is to find the first harmonic  $\omega_0$ . As in this case, it can be chosen simply as the common factor of the source frequencies, which however in some cases could require a very finely resolved spectrum and lead to an inefficient

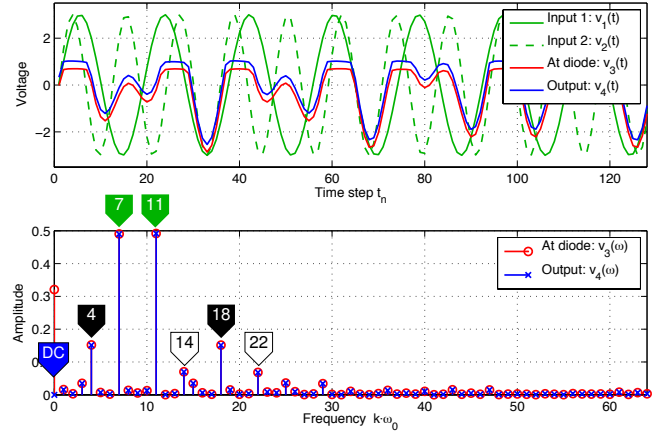


Fig. 11: Diode as mixer: time signals and spectra.

solution. Luckily there are some special techniques available, which do allow better and more efficient solutions. [4], [10]

As in previous cases, also in this case source stepping greatly improved convergence. For more complex problems it could be helpful to source-step the excitations separately (for example if there are other nonlinearities only acting on one of the excitations), but this was not further investigated in this project.

#### D. Oscillator

As last example, the well-known *Van der Pol oscillator* is introduced. Instead of a diode it contains a nonlinearity in form of a resistor with a voltage-dependent conductivity:

$$g(v) = \frac{v^2}{3} - 1 \quad (35)$$

$$i(v) = v \cdot g(v) = \frac{v^3}{3} - v \quad (36)$$

Thus, the resistor has a negative conductance around for small  $v$ . The total oscillator is a parallel circuit of this resistor, a capacitor and an inductor, as shown in Fig. 12.

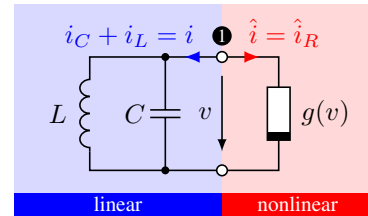


Fig. 12: Simple circuit consisting of three parts.

According to Kirchhoff's current law follows

$$i_L + i_C + \hat{i}_R = 0 \quad (37)$$

with the current/voltage relations of the capacitor

$$i_C = C\dot{v} \quad (38)$$

and inductor

$$v = L\dot{i}_L. \quad (39)$$

Using the KCL also follows  $i_L = -(i_C + \hat{i}_R)$  for the inductor current. Filling in the currents through the resistor and the capacitor and filling everything into Eqn. (39) follows the differential equation

$$v = -L \frac{d}{dt} (i_C + \hat{i}_R) = -LC\ddot{v} - \dot{v}g(v) - v \frac{\partial g(v)}{\partial v} \dot{v} \quad (40)$$

where the rule from calculus

$$\frac{dg(v)}{dt} = \frac{\partial g(v)}{\partial v} \cdot \frac{\partial v}{\partial t} \quad (41)$$

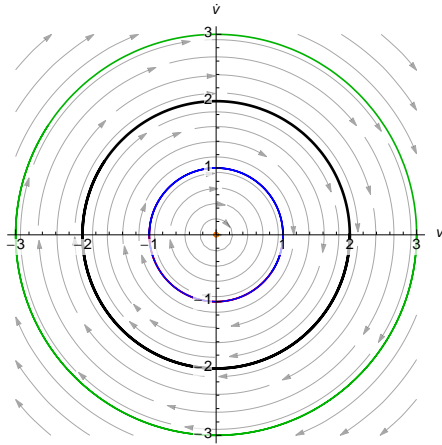
was used. By taking the derivative of the current function and rearranging the terms follow the so-called *Van der Pol differential equation*

$$LC\ddot{v} + L(v^2 - 1)\dot{v} + v = 0$$

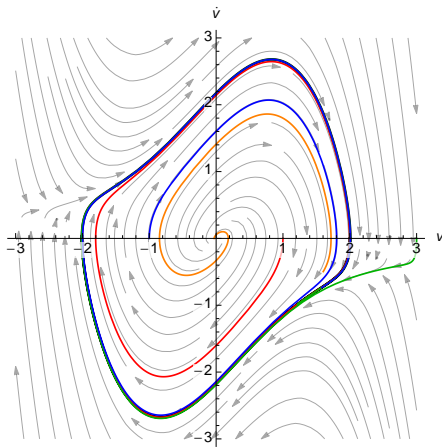
$$\ddot{v} + \epsilon(v^2 - 1)\dot{v} + v = 0$$

with  $\epsilon = 1/C = L$  for the later (more mathematical, [9]).

Note that for  $\epsilon = 0$  (and non-zero initial conditions) this becomes the normal harmonic oscillator equation with sinusoidal solutions. For  $\epsilon \neq 0$ , this leads to the well-known limit cycle attractor problem, as illustrated in Fig. 13(b).



(a)  $\epsilon = 0$  (linear): no attractors



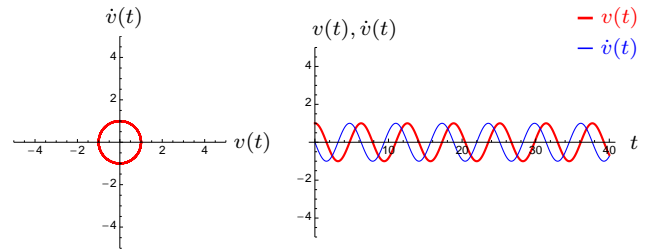
(b)  $\epsilon = 1$  (nonlinear): attracting limit cycle

**Fig. 13:** Circular force field and attractors for common harmonic oscillator and Van der Pol oscillator

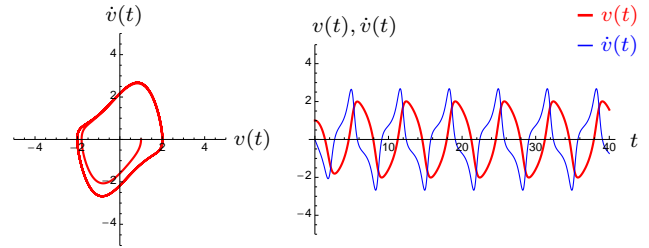
A simple figurative interpretation is as follows: in the linear harmonic case (Fig. 13(a)) there is a "tornado" at the origin

of a plane surface which rotates everything around the axis normal to this document. Thus, for example an initial value at 2 will rotate around that axis in circular motion and eventually end up at the same spot again. When the abscissa denotes the voltage, then the ordinate denotes the derivative thereof, leading to the common cosine and sine functions over time (as the tornado rotates).

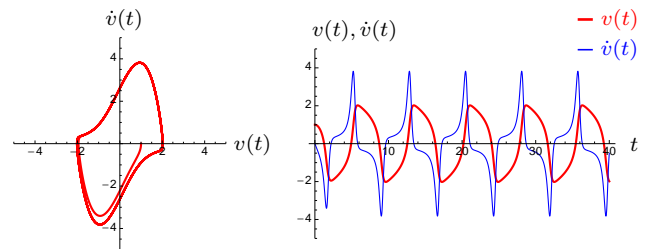
In the nonlinear case the tornado is still present, but over a hilly region. There is a hill in the middle of the domain, a (non-circular but closed) valley around it and then hills on the other side, all around the center hill. A point somewhere on the hill will be rotated due to the tornado, but will also go down the mountain (leading to a generally non-circular motion, compared to the origin). Eventually all points will end up in the valley and keep traveling in periodic but non-circular fashion. The abscissa again denotes the voltage, whereas the ordinate is its derivative. The resulting voltage forms are illustrated in Fig. 14.



(a)  $\epsilon = 0$  (linear)



(b)  $\epsilon = 1$  (somewhat nonlinear)



(c)  $\epsilon = 2$  (strongly nonlinear)

**Fig. 14:** Voltages and derivatives of the Van der Pol oscillator, compared to the common harmonic oscillator ( $\epsilon = 0$ ).

When trying to simulate this oscillator, the main problem is that the oscillating frequency cannot be calculated in closed form and the approximations are only accurate enough if  $\epsilon$  is small (i.e. the circuit just weakly nonlinear). [12]

Thus, we can only take a guess (i.e. the frequency of the harmonic oscillator) and use a  $K$  large enough. Fig. 15



shows the results obtained from MNA and HB simulations. Both reveal some problems: since the spectrum should be discrete (since the function is periodic), the MNA result is somewhat inaccurate; most likely because there were not enough periods simulated and the steady-state has not been reached. The HB method resulted in a very spiky spectrum; the voltage appears to be somewhat too smooth whereas the derivative appears to be not smooth enough. Also, since the oscillation (if there is any) always occurs at the fundamental frequency  $\omega_0$ , but it is not defined what this actually is, the frequency cannot be obtained this way. Also in this case there

#### IV. SUMMARY & CONCLUSIONS

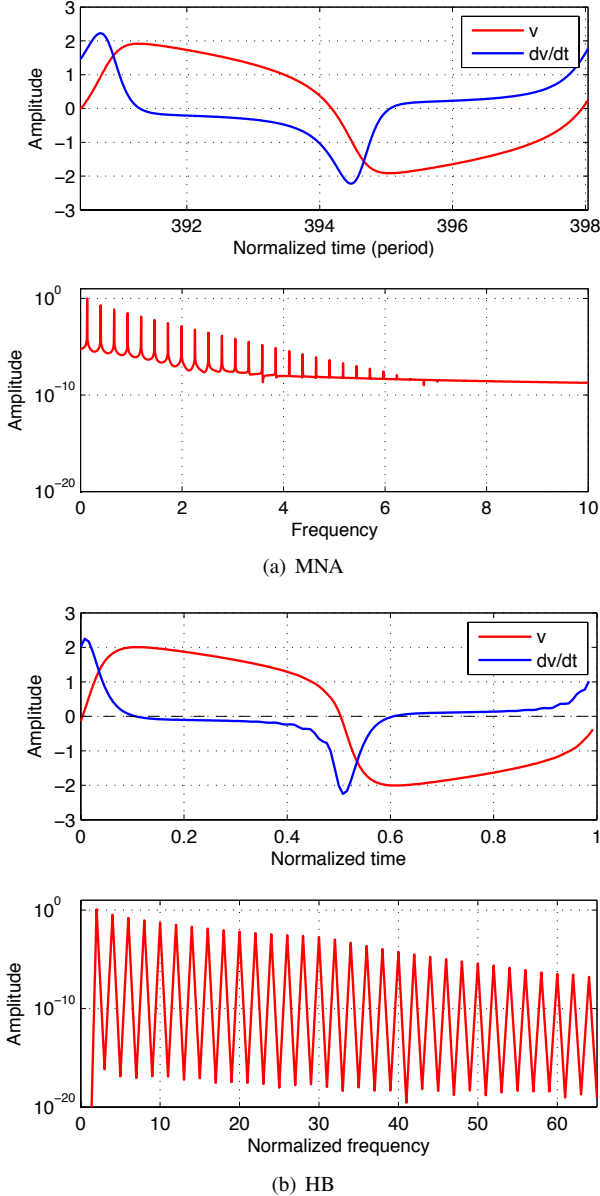
When it comes down to the question: time- or frequency-domain? The answer for microwave and RF circuits is: both. The harmonic balance method operates in both domains

- linear subcircuits  $\rightarrow$  frequency domain
- nonlinear subcircuits  $\rightarrow$  time domain
- balance currents at interfaces

(it can therefore be considered a hybrid method) and is a powerful instrument to simulate circuits containing nonlinearities as well as stiff dynamics.

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**Fig. 15:** Oscillator simulations.

are special techniques [4], [9], [10] to deal with all these problems occurring when dealing with autonomous problems. Essentially, the frequencies have to be added to the unknowns and additional equations have to be put in place to solve for them.